

Naked Singularities in Higher Dimensional Gravitational Collapse

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Spherically symmetric inhomogeneous dust collapse has been studied in higher dimensional space-time and the factors responsible for the appearance of a naked singularity are analyzed in the region close to the centre for the marginally bound case. It is clearly demonstrated that in the former case naked singularities do not appear in the space-time having more than five dimension, which appears to a strong result. The non-marginally bound collapse is also examined in five dimensions and the role of shear in developing naked singularities in this space-time is discussed in details. The five dimensional space-time is chosen in the later case because we have exact solution in closed form only in five dimension and not in any other case.

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I. INTRODUCTION

The Cosmic Censorship Conjecture [1,2] is yet one of the unresolved problems in General Relativity. It states that the space-time singularity produced by gravitational collapse must be covered by the horizon. However the singularity theorems as such do not state anything about the visibility of the singularity to an outside observer. In fact the conjecture has not yet any precise mathematical proof. Several models related to the gravitational collapse of matter has so far been constructed where one encounters a naked singularity [3-8].

It has recently been pointed out by Joshi et al [9] that the physical feature which is responsible for the formation of naked singularity is nothing but the presence of shear. It is the shear developing in the gravitational collapse, which delays the formation of the apparent horizon so that the communication is possible from the very strong gravity region to observers situated outside. Joshi et al have analyzed in details how the presence of shear determines the growth and evolution of an inhomogeneous dust distribution represented by four dimensional Tolman-Bondi metric and have attempted to clarify the nature of singularities as the final outcome.

The objective of this paper is to fully investigate the situation in the background of higher dimensional space-time with arbitrary dimensions for marginally bound collapse and in five dimensional space-time for non-marginally bound case. The reason for confining our discussions to five dimensions only in the second case is that the closed form solutions for non-marginally bound collapse are available only in $5D$ space-time. These discussions are of adequate relevance in the context of recent interest generated in the study of gravitational collapse in higher dimensional space-time [10-16].

We have shown in the present paper that for marginally bound collapse that is for $f = 0$ the naked singularity may appear only when the space-time has dimensions upto five which appears to a strong result. In more than five dimensions $t_{ah} < t_0$ that is the apparent horizon at any point in the region forms earlier than the central shell focusing singularity, which indicates that

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the shell focusing singularity first appearing at $r = 0$ remains hidden behind the apparent horizon and so gives rise to a black hole always.

The non-marginally bound collapse of a $5D$ inhomogeneous dust reveals that there may occur both the black hole and the naked singularities depending on the initial data. The five dimensional space-time is chosen in the later case because we have exact solution in closed form only in five dimension and not in any other case.

II. HIGHER DIMENSIONAL INHOMOGENEOUS DUST

The higher dimensional Tolman-Bondi type metric is given by

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - R^2 d\Omega_n^2 \quad (1)$$

where ν, λ, R are functions of the radial co-ordinate r and time t and $d\Omega_n^2$ represents the metric on the n -sphere. Since we assume the matter in the form of dust, the motion of particles will be geodesic allowing us to write

$$e^\nu = 1 \quad (2)$$

Using comoving co-ordinates one can in view of the field equations [11] arrive at the following relations in $(n + 2)$ dimensional space-time

$$e^\lambda = \frac{R'^2}{1 + f(r)} \quad (3)$$

and

$$\dot{R}^2 = f(r) + \frac{F(r)}{R^{n-1}}, \quad (4)$$

where the function $f(r)$ classifies the space-time [2] as bound, marginally bound and unbound depending on the range of its values which are respectively

$$f(r) < 0, \quad f(r) = 0, \quad f(r) > 0. \quad (5)$$

The function $F(r)$ can be interpreted as the mass function which is related with the mass contained within the comoving radius r .

The above models are characterized by the initial data specified on the initial hypersurface $t = t_i$, from which the collapse develops. As it is possible to make an arbitrary relabeling of spherical dust shells by $r \rightarrow g(r)$, without loss of generality we fix the labeling by the choice $R(t_i, r) = r$, so that initial density distribution is given by

$$\rho_i(r) = \frac{nF'}{2r^n} \quad (6)$$

Now the integration of the equation (4) gives us the exact solution for R as a function of r and t . For the case $f = 0$ the integration is straightforward to yield the solution

$$R = \left(\frac{n+1}{2} \right)^{\frac{2}{n+1}} F^{\frac{1}{n+1}} (t_s - t)^{\frac{2}{n+1}} \quad (7)$$

where t_s is a function of the radial co-ordinate r and $t = t_s$ is the instant of shell focusing singularity occurring at r , that is $R(t_s(r), r) = 0$. It indicates that the shell focusing singularity

occurs at different r at different epochs. So the collapse in this case is not simultaneous in comoving co-ordinates and is in variance with that in the homogeneous dust model [6,7,12]. The second case is for $f \neq 0$. One can not, however, obtain the explicit solution of (4) in closed form except for $n = 3$ that is in five dimensional space-time. Even in the simplest case of 4 dimensions the solution for non-zero $f(r)$ is obtained in the parametric form only.

III. marginally bound case ($f = 0$) in $(n + 2)D$ space-time

We observe that the normal vector to the boundary of any hypersphere $R - R_0 = 0$ is given by $l_\mu = (\dot{R}, R', 0, 0, 0, \dots)$. In order that l_μ is a null vector and the boundary of this hypersphere is a null surface we must have

$$R(t_{ah}, r) = [F(r)]^{\frac{1}{n-1}} \quad (8)$$

where $t = t_{ah}(r)$ is the equation of apparent horizon which marks the boundary of the trapped region. If the apparent horizon develops earlier than the time of singularity formation, the event horizon can fully cover the singularity, which then may be said to be hidden within a black hole. Combining (7) and (8) along with the initial condition $R(t_i, r) = r$ we arrive at the relation

$$(t_{ah} - t_i) = \frac{2}{n+1} r^{\frac{n+1}{2}} F^{-1/2} - \frac{2}{n+1} F^{\frac{1}{n-1}} \quad (9)$$

Since our study is restricted to the region near $r = 0$, the mass function $F(r)$ should vanish exactly at $r = 0$ and it is possible to express $F(r)$ as a polynomial function of r near the origin. One should at the same time keep in mind that initially the central density is regular. Further from physical considerations one may argue that $\rho'_i(r)$ should vanish exactly at $r = 0$, but is negative in the neighbouring region. All these consideration lead to the following expressions for $F(r)$ and the initial energy density ρ_i ,

$$F(r) = [F_0 r^{n+1} + F_2 r^{n+3} + F_3 r^{n+4} + \dots] \quad (10)$$

and

$$\rho_i(r) = \frac{n}{2} [(n+1)F_0 + (n+3)F_2 r^2 + \dots] \quad (11)$$

Obviously the initial central density is given by $\rho_c = \frac{n(n+1)}{2} F_0$. Since $\rho'_i(r) < 0$ in the region $r \approx 0$ one must have $F_2 < 0$.

Now in view of (7) one can write

$$t_s - t_i = \frac{2}{(n+1)} \frac{r^{(n+1)/2}}{[F_0 r^{n+1} + F_2 r^{n+3} + \dots]^{1/2}} \quad (12)$$

If we now denote $t_s = t_0$ as the instant of first shell focusing singularity to occur at the centre $r = 0$, the relation (12) can be used to obtain the following relation

$$t_0 - t_i = \frac{2}{(n+1)F_0^{1/2}}, \quad (13)$$

which in combination with the relation (9) yields

$$t_{ah} - t_0 = - \left[\frac{1}{n+1} \frac{F_2}{F_0^{3/2}} r^2 + \frac{1}{n+1} \frac{F_3}{F_0^{3/2}} r^3 + \frac{1}{n+1} \frac{F_4}{F_0^{3/2}} r^4 + \dots \right]$$

$$- \left[\frac{2}{n+1} F_0^{\frac{1}{n-1}} r^{\frac{n+1}{n-1}} + \frac{2}{n^2-1} \frac{F_2}{F_0^{\frac{n-2}{n-1}}} r^{\frac{3n-1}{n-1}} + \frac{2}{n^2-1} \frac{F_3}{F_0^{\frac{n-2}{n-1}}} r^{\frac{4n-2}{n-1}} + \dots \right] \quad (14)$$

The relation (14) is valid in general $(n+2)$ dimensional space-time and reduces to the 4 dimensional expression ($n=2$) in the region near the centre as given in the paper of Joshi et al [9]. In $5D$ space-time the naked singularity appears only if $F_2 \neq 0$ and further $|F_2| > 2F_0^2$. Otherwise for either $F_2 = 0$ or $|F_2| < 2F_0^2$ there must occur a black hole, because in that case $t_{ah} < t_0$ that is, the horizon appears earlier than the shell focusing singularity at $r=0$. Only in a very special case of $|F_2| = 2F_0^2$ the occurrence of naked singularity or black hole will depend on the co-efficients of higher powers of r . For general $(n+2)$ dimensions the close examination of (14) reveals that if there is to exist a naked singularity we must have

$$\frac{n+1}{n-1} \geq 2, \quad (15)$$

which means $n \leq 3$. It is interesting to note that $\left(\frac{n+1}{n-1}\right)$ has the value 2 when $n=3$ and then it decreases monotonically with the increasing number of dimensions. So for space-time with dimension larger than five the term $\frac{2}{n+1} F_0^{\frac{1}{n-1}} r^{\frac{n+1}{n-1}}$ dominates leading to a black hole. Our conclusion is that for the space-time having larger than five dimensions the existence of the naked singularity is prohibited or in other words the shell focusing singularity is fully covered by the horizon in such cases. However, if we relax the restriction namely, $\rho'_i(r) = 0$ at $r=0$ then F_1 will be non-zero and this will lead to the possibility of naked singularity in any dimension.

Calculation of shear near $r=0$:

We still consider the marginally bound case $f=0$. The shear scalar in the $(n+2)$ dimensional spherically symmetric dust metric so far discussed in this section may be estimated by the factor [9] $\sigma = \sqrt{\frac{n}{2(n+1)}} \left(\frac{\dot{\lambda}}{2} - \frac{\dot{R}}{R} \right)$. Using (3) and (4) one gets in turn

$$\sigma = \sqrt{\frac{n}{2(n+1)}} \left(\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R} \right) = \sqrt{\frac{n}{8(n+1)}} \frac{[RF' - (n+1)R'F]}{F^{1/2}R^{(n+1)/2}R'} \quad (16)$$

From (7)

$$R' = \left[\frac{n+1}{2} \right]^{\frac{2}{n+1}} \left[\left(F^{\frac{1}{n+1}} \right)' (t_s - t)^{\frac{2}{n+1}} + \frac{2}{n+1} F^{\frac{1}{n+1}} (t_s - t)^{\frac{2}{n+1}-1} t'_s \right]$$

where F, F', t_s etc.'s are all functions of the comoving radial co-ordinate r . So as t approaches t_s , R approaches zero and effectively σ^2 approaches $(n+1)^2 \frac{F}{R^{n+1}}$, which in turn becomes infinitely large. It is therefore evident that at each r , the magnitude of shear explodes as the energy density explodes ($R=0$) and this occurs at different instants at different spherical shells of different radial co-ordinates including the centre ($r=0$).

It is now possible in view of the choice $R(t_i, r) = r$ to express the initial shear σ_i to the form

$$\sigma_i = \sqrt{\frac{n}{8(n+1)}} \frac{[rF' - (n+1)F]}{F^{1/2}r^{(n+1)/2}} \quad (17)$$

But using (7) along with the initial condition $R(t_i, r) = r$, we have a relation like

$$t - t_i = F^{-1/2} \left[r^{\frac{n+1}{2}} - R^{\frac{n+1}{2}} \right] \quad (18)$$

which in turn being used in (16) yields the expression for shear scalar $\sigma(t, r)$ very near the centre ($r \approx 0$) in the following form after some manipulations:

$$\sigma = \sqrt{\frac{n}{8(n+1)}} \frac{\sum_{m=2}^{\infty} m F_m r^m}{F_0^{1/2} \left[1 + \frac{(n+1)^2}{4} F_0 (t - t_i)^2 - (n+1) F_0^{1/2} (t - t_i) \right]} \quad (19)$$

so the initial shear at $t = t_i$ is given by

$$\sigma_i = \sqrt{\frac{n}{8(n+1)}} \frac{\sum_{m=2}^{\infty} m F_m r^m}{F_0^{1/2}} \quad (20)$$

We find that at the centre ($r = 0$) the initial shear vanishes. It is interesting to note that the dependence on r of the initial shear does not depend on the number of dimensions of the space-time. In fact the expression (20) coincides exactly with the value of the initial shear σ_i calculated earlier by Joshi et al [9] for $4D$ space-time ($n = 2$). In view of what has been discussed so far the expression (20) reveals that the existence of the naked singularity is directly related with the non-vanishing shear in four and five dimensions. But in larger dimensions the central singularity seems to be covered by the appearance of apparent horizons. One should mention here that the statement of the paper of Joshi et al [9] that the shear decreases in course of time and finally vanishes at $t = t_0$ is not true. In fact the shear in view of (19) increases and goes to infinity at this epoch, which is expected.

IV. NON-MARGINALLY BOUND FIVE DIMENSIONAL SPACE-TIME ($f \neq 0$)

We already know that for $f \neq 0$ it is possible to obtain the solution of (4) in closed form only in $5D$ space-time that is for $n = 3$. In other cases even for the simplest case of a four dimensional manifold the solution is available only in parametric form. In $5D$ space-time one of the exact solution of the equation (4) is given by

$$R^2 = \left[f(t_s - t)^2 + 2F^{1/2}(t_s - t) \right] \quad (21)$$

For $f = 0$ the solution (21) is coincident with the solution (7) when we put $n = 3$. By the same arguments put forward in the previous section it is possible to express $F(r)$ as

$$F(r) = F_0 r^4 + F_2 r^6 + F_3 r^7 + \dots \quad (22)$$

and the initial density distribution $\rho_i(r)$ as

$$\rho_i(r) = \rho_c + \rho_2 r^2 + \rho_3 r^3 + \dots, \quad (23)$$

where and $\rho_c = 6F_0$, $\rho_2 = 9F_2$, so on.

Here also following the earlier reasoning $F_2 < 0$. The only new input in the present case is the function $f(r)$, which may be expressed as a power series in r near the centre $r = 0$.

We assume

$$f(r) = f_0 r + f_1 r^2 + f_2 r^3 + \dots \quad (24)$$

The form (24) is chosen because $f(r)$ vanishes as $r \rightarrow 0$, which is demanded by the regularity condition at $r = 0$ [17].

Now when $t = t_i$ one can write from (21)

$$t_s - t_i = \frac{(F + fr^2)^{1/2}}{f} - \frac{F^{1/2}}{f} \quad (25)$$

The finite non zero magnitude of the left hand side of (25) demands $f_0 = 0$ and $f_1 \neq 0$ as minimum requirements. The expansion of (22) near $r = 0$ yields

$$\begin{aligned} t_s - t_i = & \frac{(F_0 + f_1)^{1/2} - F_0^{1/2}}{f_1} + \frac{f_2}{f_1^2} \left[\frac{2F_0^{1/2}(F_0 + f_1)^{1/2} - (2F_0 + f_1)}{2(F_0 + f_1)^{1/2}} \right] r + \left[-\frac{f_2^2}{2f_1^2\sqrt{f_1 + F_0}} \right. \\ & \left. + \frac{(f_2^2 - f_1f_3)(\sqrt{f_1 + F_0} - \sqrt{F_0})}{f_1^3} - \frac{F_2}{2f_1\sqrt{F_0}} + \frac{4(f_1 + F_0)(f_3 + F_2) - f_2^2}{8f_1(f_1 + F_0)^{3/2}} \right] r^2 + \dots \quad (26) \end{aligned}$$

At the centre $r = 0$ the relation (26) reduces exactly to

$$t_0 - t_i = \frac{(F_0 + f_1)^{1/2} - F_0^{1/2}}{f_1} \quad (27)$$

Now at $t = t_{ah}$ one must have $R^2 = F$ as justified earlier. Hence by solving (21) we get

$$t_{ah}(r) - t_s(r) = \frac{F^{1/2} - F^{1/2}(1 + f)^{1/2}}{f} \quad (28)$$

The form (28) is chosen because it satisfies the consistency relation in the limit $f \approx 0$. Remembering that $t_s(r = 0) = t_0$ we obtain in the region very close to the centre

$$\begin{aligned} t_{ah}(r) = t_0 + & \frac{f_2}{f_1^2} \left[\frac{2F_0^{1/2}(F_0 + f_1)^{1/2} - (2F_0 + f_1)}{2(F_0 + f_1)^{1/2}} \right] r + \left[-\frac{f_2^2}{2f_1^2\sqrt{f_1 + F_0}} - \frac{f_1F_0 + F_2}{2f_1\sqrt{F_0}} \right. \\ & \left. + \frac{(f_2^2 - f_1f_3)(\sqrt{f_1 + F_0} - \sqrt{F_0})}{f_1^3} + \frac{4(f_1 + F_0)(f_3 + F_2) - f_2^2}{8f_1(f_1 + F_0)^{3/2}} \right] r^2 + \dots \quad (29) \end{aligned}$$

In the above relation if $f_2 < 0$ the second term on the R.H.S is positive, which can easily be verified. Hence it is possible to conclude that $t_{ah} > t_0$ near $r = 0$. This suggests that the apparent horizon in the region close to the centre appears later than the shell focusing singularity at $r = 0$ and hence the naked singularity should exist in this case. On the other hand if $f_2 > 0$ it should be a black hole. These conclusions are valid irrespective of $f(r)$ be positive or negative except for the restriction $F_0 > |f_1|$ in case $f_1 < 0$. However, no definite conclusion can be drawn in case $f_2 = 0$.

Calculation of shear:

The general expression for the shear in five dimensional space-time is given by

$$\sigma = \sqrt{\frac{3}{32}} \left(f + \frac{F}{R^2} \right)^{-1/2} \left[\frac{f'}{R'} + \frac{F'}{R^2 R'} - \frac{4F}{R^3} - \frac{2f}{R} \right] \quad (30)$$

Using the expansions of $F(r)$ and $f(r)$ from (22) and (24) along with the assumption $f_0 = 0$, which has already been justified earlier we arrive at the expression for the shear at any instant in the region close to $r = 0$. This expression is given after omitting a few intermediate steps by

$$\sigma = \sqrt{\frac{3}{32}}(F_0 + f_1)^{-1/2} \left[\frac{f_2}{X} r + m \left(\frac{F_m}{X^3} + \frac{f_{m+1}}{X} \right) r^m + O(r^{m+1}) \right] \quad (31)$$

where $m \geq 2$ and $X = X(t) = [1 + f_1(t - t_i)^2 - 2(F_0 + f_1)^{1/2}(t - t_i)]$.

Since σ depends on the time the shear develops in course of time at any r away from the centre. When $f = 0$ we get for the initial shear at $t = t_i$

$$\sigma = \sqrt{\frac{3}{32}} \frac{mF_m}{F_0^{1/2}} r^m + O(r^{m+1})$$

with $m \geq 2$. This is the same expression as in 4D case.

It is evident that the initial shear vanishes when $f_2 = 0, (F_2 + f_3) = (F_3 + f_4) = \dots = 0$ and hence even if the initial shear is zero the dust distribution may be inhomogeneous because the co-efficient F_2 may still be non-zero. This makes in view of (26) t_s a function of the comoving radial co-ordinate r , so that the shell focusing singularity appears at different r at different instants. The nature of singularities appearing in marginally bound cases ($f = 0$) is therefore clearly distinct from the present case of non-marginally bound case ($f \neq 0$). This fact, however, has not been taken care of by Joshi et al in their discussion in the 4 dimensional marginally bound space-time.

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